..." Dr. Rodden insisted that "solution procedure" should be read as "a new solution." Thus, the author's intention of developing a finite element procedure was misinterpreted as claiming a new solution. By twisting such key words, Dr. Rodden found himself a chance to tell a long story in the *AIAA Journal* about his Ph.D. thesis.

Dr. Rodden asked "What's new?" between the finite element displacement method the author used and the kind of old, obsolete flexibility influence coefficient method he used. The answers are as follows:

1) The finite element displacement model in this paper has displacement and slope degrees-of-freedom at nodal points and provides a 4×4 stiffness matrix. In the examples, four elements and six degrees of freedom are used. Dr. Rodden included only the displacements at the nine collocation control points, neglected the important slope degrees of freedom, and formulated the 9×9 flexibility matrix for the ten-segment span. Although Dr. Rodden claimed that his approach was the finite element method, clearly it was not. Not all the matrix methods are finite element methods!

2) In this paper, the consistent mass matrix is used. Dr. Rodden used a lump mass diagonal matrix. The obvious difference in the resulting accuracy between the consistent mass and the lumped mass diagonal matrices for the case of beams or infinite plates was pointed out by Archer (Ref. 13 in the Comment). In the dynamic eigenvalue problems of beams or infinite plates, it is possible that one can use an inferior flexibility matrix and an inferior lumped mass matrix simultaneously and be satisfied with the results due to the compensation of modeling errors.

3) The incremental stiffness matrix in this paper can accurately account for the important effect of initial in-plane stresses. This point is studied by the author. Examples are performed and results are presented in Figs. 6-8. Such effect was not considered by Dr. Rodden in his thesis.

4) The beauty of a numerical method does not necessarily lie in its sophistication. When the simple trapezoidal rule can achieve excellent accuracy in approximating the aerodynamic pressure, the use of sophisticated higher-order numerical integration method is of no value, especially when the structural model is crude.

Since their appearance in 1956[1] the finite element methods have sometimes been criticized for the emphasis on the methodology rather than new theory. The methods have gradually gained widespread acceptance because of their powerfulness in solving the practical problems which cannot be solved otherwise. In the development of each new finite element method, it is necessary to choose some examples with known solutions for comparison and evaluation. Once the method is evaluated, it can be used for more general and practical cases. In this paper, Cunningham's examples and solution in Ref. 7 were chosen for such an evaluation purpose. Reference 7 was published earlier than Dr. Rodden's thesis. It is absolutely pointless to reference a thesis later published by Dr. Rodden which provides the same solution as Ref. 7.

This paper establishes a basic procedure of extending the finite element method (displacement models) to include the aerodynamic effects for flutter analysis. Previous similar attempts have been made by Olson (Refs. 1 and 2 of original paper), Kariappa et al.[2] (Refs. 3 and 4 of original paper), and Sander et al.[2] The basic development in this paper has recently been extended to include the effect of thermal buckling and geometric nonlinearlity[3] and three-dimensional supersonic unsteady potential flow.[4] Contrary to a footnote in the Comment, it has been shown in Ref. 4 that the computing expense is not prohibitive in using the supersonic Mach box method.

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Comment on "Localized Diamond-Shaped Buckling Patterns of Axially Compressed Cylindrical Shells"

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I READ with great interest several articles on the isometric buckling of shells.[1] However, I am unable to trace how the vital coefficients K_I, K_I are derived so perhaps El Naschie could give the derivation of these coefficients in detail. This would be of great help to the reader.

I would also like to ask about the connection to the work of Yoshimura.[2] El Naschie does not refer to this work although it seems to me to deal with similar ideas.

Finally, great interest has been awakened in an engineering approach to shell buckling along similar lines since the publication of El Naschie's first work. This is mainly due to the recent works of Edlund,[3] Fritz and Wittek,[4] and Croll.[5] Perhaps El Naschie could comment on these works and their interrelationship. Such a comparison would help to lessen the confusion arising from the numerous shell buckling theories.

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Reply by Author to P. Kaoulla

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I WOULD like to thank P. Kaoulla for his interest in the work and for his relevant remarks. As for the derivation of

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K_1 and K_2, these were not given explicitly mainly due to lack of space. The simplest way of deriving these coefficients is as follows.

Let the length of an imaginary isolated longitudinal element of a cylindrical shell be ℓ and the moment of inertia be I. This element is elastically supported by the bending stiffness of imaginary rings in the circumferential direction of the cylinder plus the torsional stiffness due to the plate effect of the, in reality, continuous cylinder. The only difference from a strut on an elastic foundation is that the moment of inertia is not simply $bt^3/12$ where t is the thickness of the shell and b is the breadth of the fictitious strut because of the curvature effect in the circumference. This curvature gives the strip considerable stiffness which should not be neglected. Consequently, we have to consider a longitudinal strip of a breadth equal to half the buckling waves in the circumferential direction. Thus, the ficitious longitudinal element in the case of an axially compressed cylindrical shell is an imaginary strut with the bending stiffness equal to that of a flat plate $[E/(1-\nu^2)]\cdot(bt^3/12)$ plus the curvature influence $[E/(1-\nu^2)]\cdot b^5t/720r^2$. Assuming a sinusoidal deformation in the circumferential direction of the form

$$w_s = w_0 \sin(\pi/b)s$$

where b is the half-waves length, the stabilizing forces acting on one column due to the bending of the circumferential rings of the breadth unity are

$$p=\int_0^b \frac{Et^3}{12(1-\nu^2)}\frac{\partial^4 w}{\partial s^4}\mathrm{d}s$$

$$=\int_0^b \frac{Et^3}{12(1-\nu^2)}\frac{\pi^4}{b^4}w_0 \sin\frac{\pi}{b}s\mathrm{d}s=\frac{\pi^3 Et^3}{6(1-\nu^2)b^3}w_0$$

It is seen that the forces are proportional to the deflection w and this corresponds to a Winkler type of linear elastic foundation where the foundation constant is

$$C=\pi^3 Et^3/6(1-\nu^2)b^3$$

There now remains the foundation constant due to the twisting of the cylinder middle surface. This can be taken over from the known plate theory and we only need to integrate over the wave length b. In this way we obtain

$$G=\frac{2Et^3}{(1-\nu^2)}\int_0^b\left(\frac{\pi}{b}\right)^2\sin\frac{\pi}{b}s\mathrm{d}s=2\left(\frac{Et^3}{6(1-\nu^2)}\frac{\pi}{b}\right)$$

Inserting C and G in the formula for the buckling load of a strut on a Pasternak foundation

$$P^c=\frac{i^2EI\pi^2}{\ell^2}+\frac{c\ell^2}{\pi^2 i^2}+G$$

we find that

$$\sigma^c\frac{P^c}{bt}=\frac{\pi^2}{6(1-\nu^2)}\left[\frac{Eb^4}{120r^2\ell^2}+\frac{Et^2}{2\ell^2}+\frac{Et^2\ell^2}{\pi b^4}+\frac{2Et^2}{\pi b^2}\right]$$

The next step is, of course, to obtain the smallest critical value by minimizing this expression. We observe that σ^c takes its smallest value when ℓ and b become very large compared with t while, simultaneously, the first and third terms become equal. The second and fourth terms can then be neglected. Physically, this means that the bending stiffness $Ebt^2/12(1-\nu^2)$ can be neglected compared with the curvature effect and second, that G can be neglected compared with C or, in other words, the torsional rigidity is negligible in the advanced postbuckling stage compared to the bending rigidity. As a result of this, the energy functional of a strut on an elastic foundation

$$V=\int_0^\rho (C_1\ddot{w}^2+C_2w^2+P\dot{w}^2)2\pi r\mathrm{d}x$$

can be used to investigate the postcritical behavior of the cylinder if we change

$$C_1=EI=\frac{E}{(1-\nu^2)}\frac{t^3}{12}$$

$$C_2=p=\frac{Et}{2r^2}$$

as follows

$$C_1\rightarrow k_1=\frac{1}{2}\left(\frac{Etb^5}{720r^2(1-\nu^2)}\right)$$

$$C_2\rightarrow k_2=\frac{E}{12}\frac{t\pi^3}{(1-\nu^2)b^3}$$

Now I come to the question concerning the work of Yoshimura. I referred to this work in my thesis[1] (pp. 234, 337; Ref. 121). It is true that I did not refer to this in some of my later works but again, this was mainly due to lack of space. However, in two of my recent papers,[2,3] one of which is in the IUTAM Symposium in Tokyo, I give in some detail a comparison between my work and that of Yoshimura. Roughly speaking, Yoshimura's work is confined to the overall buckling configuration of a cylindrical shell. His buckling load is only half of the classical buckling load. My work is mainly an extension of Kirste's and Pogorelov's work and deals with localized buckling forms which are more in agreement with experimental observations. The buckling load obtained is also much smaller (about one quarter of the classical value) and is, therefore, of much more practical value.

Finally, as to the last question, concerning the connection to the work of Edlund, Fritz and Wittek, and Croll, I agree that a correlation study between the different buckling theories is most vital and that the picture at the present time is somewhat confusing. It is, of course, very difficult to state in a few lines the essence of the work of these authors. However, roughly speaking, all this work deals with the same basic idea, which is inextensional deformation. Nevertheless, there are some vital differences, and for this reason we might divide the work done by the inextensional school into three groups.

The first consists mainly of the work of Pogorelov, Ashwell, Yoshimura, Kirste, Edlund, and myself. I think that the work of this group is consistent with the other well established theories of von Karman and Koiter. The second group is that of Fritz and Wittek, who simply ignore the membrane energy in the energy functional. I have repeated the calculation for a cylindrical shell under axial pressure using their theory and obtained as a buckling stress

$$\sigma^c=0.6\frac{Et^2}{(1-\nu^2)r^2}$$

That means the buckling stress is very small and is a function of $(t/r)^2$. This contradicts the results of an accurate nonlinear analysis where it is found to be function of t/r. I have, therefore, some doubt about the practical value of this method. The third group mainly consists of the work of Croll and his students. Croll attributes imperfection sensitivity to the membrane strain energy. He, therefore, ignores the contribution of the membrane force to the buckling load. For instance, as the classical buckling load of a cylindrical shell under axial pressure contains about 50% membrane stress, he concludes that $\sigma^c = \frac{1}{2}\sigma^c_{(classical)}$ is a lower bound estimate and rules out the influence of boundary conditions. However, I have calculated the membrane contribution for the critical load of the free edge cylindrical shell following Croll and found that according to his theory a lower bound must be $\sigma^c = \frac{1}{4}\sigma^c_{(classical)}$ and not $\sigma^c = \frac{1}{2}\sigma^c_{(classical)}$. It is also interesting to note that imperfection sensitivity in some axially inextensional structures can exist due to nonlinear compatibility conditions and regardless of the amount of membrane strain energy as pointed out in detail in Refs. 4 and 5.

Having said that, we must, of course, admit that the works of the second and third groups have, nevertheless, illuminated many dark corners and clarified for us many ambiguous points.

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Comment on "Inviscid Hypersonic Flow around a Semicone Body"

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KIMURA and Tsutahara propose in Ref. 1 a solution for the inviscid hypersonic flow past a flat topped semicone body. Having established by an unsteady analogy that the cross-flow velocity components obey the Laplace equation they proceed to solve this under boundary conditions which include an assumption of vanishing velocities at infinite radius. They then combine the known density ratio across shock waves at infinite Mach number with the continuity equation to obtain the shock shape from the calculated velocities. Since this interposes a discontinuity between the body and the boundary condition at infinity, one is inclined to suspect the relevance of that condition. In fact, that boundary condition can only be satisfied because a genuinely relevant condition has been omitted. Consider a conical shockwave $r/x = r_s(\theta)$, normal to which a vector in (x,r,θ) coordinates may be written as $n = (r_s, -1, r_s^{-1} dr_s/d\theta)$. The equation for continuity of mass may be written $\rho_1 (q_1 \cdot n) = \rho_2 (q_2 \cdot n)$ or

$$v_\theta \frac{dr_s}{d\theta} + r_s^2 \left[v_x - \frac{\gamma - 1}{\gamma + 1} U \right] - v_r r_s = 0 \tag{1}$$

The equation for continuity of momentum parallel to the shock is

$$q_1 \times n = q_2 \times n$$

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which yields two independent equations

$$U - v_x - v_r r_s = 0 \tag{2}$$

$$v_r \frac{dr_s}{d\theta} + v_\theta r_s = 0 \tag{3}$$

Equation (2) may be interpreted as $v_x = U + 0(r^2/x^2)$, which is intuitive anyway, and combined with Eq. (1) to yield

$$v_\theta \frac{dr_s}{d\theta} + \left[1 - \frac{\gamma - 1}{\gamma + 1} \right] r_s^2 - v_r r_s = 0 \tag{4}$$

which is Eq. (10) of Ref. 1, and is employed there to find the shock shape. Equation (3), however, is not considered at all, although it has quite equal status with Eq. (4) as an identity which must be satisfied on the shock. The condition for (3) and (4) to be compatible is

$$v_r^2 + v_\theta^2 = 2 v_r r_s / (\gamma + 1) \tag{5}$$

It is this boundary condition, to be applied on the initially unknown shock-wave, which should replace the spurious boundary condition at infinity. It should be noted that Eq. (3), and hence Eq. (5), is automatically satisfied in axisymmetric flow ($v_\theta = 0$), or two-dimensional flow ($v_\theta = v_r \tan \theta$), and it would seem safe to neglect it in situations close to one of these. However, inspection of the results of Ref. 1 indicated a large region, roughly $\pi/4 < |\theta| < \pi/2$, where neither assumption is satisfied. Thus, although the flow in $|\theta| < \pi/4$ may be self-consistent, it cannot be regarded as caused by removing the cone top, since the regions in which cause and effect operate are separated by a region in which the flow model breaks down.

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Reply by Authors to P. L. Roe

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IT is natural that the condition for Eq. (3) in Roe's comment must be considered. However such momentum relations are sometimes neglected in constant density solutions.[1] For a complete solution, the relations between the flows of the two sides, before and behind the shock wave, such as the conservation equation for energy and the relations of entropy change, must be satisfied, as well as Eq. (3). Since the strength of the shock wave cannot be the same everywhere, they are contradictory to our assumption that the internal energy and entropy are uniform within the shock layer. Therefore we do not think that the previous assumption leads a satisfactory solution for the whole flowfield.

Mr. Roe says that it is Eq. (5) in his Comment, to be applied on the initially unknown shock wave, which should replace our spurious boundary condition at infinity. However, it is difficult to think that this will improve our analysis, because the contradictions in the region where θ is large are essentially

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